

马可夫连锁模型对信用风险商品之评价

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摘要

本研究的目的在于扩展 Jarrow 等 (1997) 的模型, 把随机无违约风险利率和随机信用差异纳入这个模型。信用差异随机程序是经由风险溢价调整项的随机程序来认定的, 而非按大家所熟知某些动态过程的假设。除了信用等级的改变之外, 这个模型也把每一个信用等级的随机信用差异列入计算, 因此, 这个模型能够考虑到构成信用差异曲线的连续或跳跃的成份。在这个模型中, 就算信用等级维持不变, 信用差异还是可以改变的。这个模型有四个特色: 第一, 信用差异和约当移转矩阵 (包括风险中立违约机率) 是由当前的信用等级和产生随机过程的其它变量所决定的。第二, 在无套利机会的状态下, 结合远期利率的程序和信用差异的间接形式可以得到远期利率程序中风险中立变动项的回归表示和违约风险结构。此外, 这个模型可以帮助各种不同的信用衍生商品定价, 也可以帮助我们利用信用等级的改变和违约的历史数据和信息。最后, 这个模型可以延伸为连续的模型, 把不同的风险溢价调整项包含进来。

关键词: 信用差异、移转矩阵、信用差异期权、信用差异互换

一、前言

金融证券的评价模型通常假设有关方面会履行契约中的责任。然而, 契约中的一方有可能没有履行责任。目前许多金融方面的研究都是有关公司违约风险的定价。此外, 关于信用衍生商品的研究也愈来愈多。这些研究发展出一些不同的模型来测定信用衍生商品的溢价, 以及描述有违约风险公司债券的结

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构。这些模型都是从 Merton (1974) 最初的模型所推论延伸而来的，这些模型被称为「信用风险的结构模型」，因为它们的假设都是建立在公司结构上。另一个方法称为「简约式模型」，因为这些模型直接处理可以从结构模型推算出来的公司债券利率结构。

信用衍生商品的定价需要信用风险的信息，因为这些商品非常容易受到公司的信用状况所影响。有违约风险的公司债券在它真正违约前能够承受几次信用评级改变，市场也会对这些改变定价。因此，过去的信用评级和违约的信息对于有违约风险的公司债券和信用衍生商品是非常有用的。Jarrow 等 (1997) 发展出的马可夫连锁 (Markov Chain) 模型把信用评级并入违约可能性的指标来描述信用风险差异的结构。

信用风险和其它风险 (如利率风险或资产风险) 有相同以及不同之处。信用风险和利率风险一样都是可以交易的；然而信用风险交易的流通性不如利率风险交易的流通性大，这是因为信用风险交易的市场比较小。利率风险和信用风险第二个不同的地方在于信用风险的变化通常会引发相关信用差异的“跳动”，特别是当这些改变是因为违约所引起的。Duffee (1999) 用总体经济的改变来描述信用差异的动态。Elton 等 (2001) 利用实证方法观察公司债券差异并把信用差异当作税、违约风险和系统风险因素的函数。Huang and Kong (2003) 研究五组变量 (违约率、利率变量、股票市场因素、流动性指标，以及总体经济指标) 对于信用差异变化的影响，证明了信用风险模型可能需要并入总体经济变量才能描述信用差异的变动。对信誉的重视和对市场风险的厌恶都指出了我们需要把信用风险变动当成一个随机程序。

本研究的目的在于扩展 Jarrow 等 (1997) 的模型，把随机无违约风险利率和随机信用差异纳入这个模型。信用差异随机程序是经由风险溢价调整项的随机程序来认定的，而非按大家所熟知某些动态过程的假设。除了信用评级的改变之外，这个模型也把每一个信用评级的随机信用差异列入计算，因此，这个模型能够考虑到构成信用差异曲线的连续或跳跃的成份。在这个模型中，就算信评维持不变，信用差异还是可以改变的。这个模型有四个特色：第一，信用差异和约当移转矩阵 (包括风险中立违约机率) 是由当前的信用评级和产生随机过程的其它变量所决定的。第二，在无套利机会的状态下，结合远期利率的程序和信用差异的间接形式可以得到远期利率程序中风险中立变动项的回归表示和违约风险结构。此外，这个模型可以帮助各种不同的信用衍生商品的定价，也可以帮助我们利用信用差异的改变和违约的历史数据和信息。最后，这个模型可以延伸为连续模型，把不同的风险溢价调整项包含进来。

除了第一部分序言外，本篇研究包含以下部分：第二部分比较了一些分析信用风险的方法，并探讨了文献中的几个模型。第三部分描述远期利率程序和约当移转矩阵，对于这个模型的背景也有详细的说明。第四部分讨论如何完成这个模型，并发展出双二项式树形图 (double binomial tree)。第五部分是应用模型来决定信用差异选择权以及信用互换的价值，以作为信用衍生商品订价的例子。第六部分则是本研究的结论。

二、文献回顾

违约风险可以分解成为两个要素：违约几率和回收比率。违约几率是在某一段特定时间中违约事件发生的可能性，回收比率则是违约债券的已付比率。对于此两种要素不同的处理方式，连同违约风险和利率风险的交互作用，引发了各种不同的研究。在文献中可以看到关于违约风险有两种不同的模型：结构模型和简约模型。

(一) 结构模型

Merton (1974) 假设公司只用一种债券，因此公司的价值会随着随机过程而变动，由此推算出违约风险债券的价值。他把股东权益视为公司价值的买权，债券的价值就是公司价值和股东权益之间差价，因此可以用 Black and Scholes (1973) 及 Merton (1973) 所发展出的选择权定价模型来推导出有违约风险的债券的价格。这个定价公式需要五项信息：公司价值、债券票面价值、公司价值的波动率、具相同到期日之无违约风险债券回报率、以及债券的到期日。尽管 Merton 的模型很简要，表面上也很具吸引力，但是这个模型也有一些限制。第一，这个模型假设只有在到期日时才会有违约的情形发生，这个假设是不实际的。第二，这个模型假设只有在公司的资产用完时才会违约，可是通常公司在资产用完之前就会有违约的情形发生了。Jones 等 (1984) and Franks and Torous (1989) 认为这种假设会使信用差异远小于实际值。第三，在公司的资本结构中通常不会只有一种债券，因此还需要详细说明各种债券的优先结构。而且，这个模型的架构假设所有公司各种权利的拥有者中会有一个优先顺位权可以用来分配公司的资产。这个模型另外还有一个问题是，公司的资产通常是不可以买卖的，也无法估算它们的价值，因此，要应用这个模型和要计算相关的变量就会有困难。

Geske (1977) 把 Merton (1974) 的分析延伸到有到期日以及时间间断的票面息支付的风险债券。Black and Cox (1976) 则是把 Merton 的模型延伸来研究契约损害赔偿、清偿顺位的安排，以及融资的限制。他们把公司只有在资产都用完时才会发生违约情形的这个假设条件放宽，他们假设当公司资产价值到达某个低水平时就会发生违约的情形。他们的模型比较实际，也符合以净值为基准或是以现金流量为基准的破产条件。他们也考虑到违约可能在到期日之前便已发生。因此这个模型所造成的信用差异和在公司债券市场所看到的是比较一致的。但是，Black and Cox 的模型对于有风险债券的估价也是有一些其它的限制的。他们假设利率是不变的，可是这样就很难用来证明有风险的固定收益债券的估价模型。他们也假设公司各种权利的拥有者中会有一个优先顺位权可以用来分配公司的资产。然而，最新的研究，如 Franks and Torous (1989, 1994)；Eberhart 等 (1990)；LoPucki and Whitford (1990)；Weiss (1990)；Betker (1995) 都显示在破产重组的情形下无法维持优先顺位权。

Turnbull (1979) 把 Merton 的模型加以推导, 加进公司税和破产成本, 进而推导出对于公司的普通股和单纯折价债券的公式解 (Closed Form Solution)。Bhattacharya and Mason (1981) 把 Merton 对于公司价值的分析做了更进一步的延伸, 他们遵循非连续性程序以及更复杂的边界条件。Kim 等 (1987) 发展出或有索取权评估模型, 这个模型中所引导出来的违约补偿和实际观察是一致的。他们假设无违约风险利率会跟随 Vasicek (1977) 程序并且和公司价值程序相关。他们在股利和利率的不确定性、以及利率风险和违约风险交互作用这两个因素存在的情形下, 研究债券票面息的违约风险。他们的研究显示了违约补偿容易受到对于利率预期的影响, 但是不会受到利率波动性的影响。他们的研究也显示了可赎回条款对于和公司相关的资金收益有不同的影响。当公司的资产价值遵守着有固定波动性的随机程序时, Leland (1994) 导出了长期风险负债、回报差异、最佳资本结构的公式解。他明确地把债券价值及最佳财务杠杆状况和公司风险、税赋、破产成本、无违约风险利率、股利的支付率、及债券条款连结在一起。这个结果解释了垃圾债券和投资等级债券的不同, 也解释了资产替代、债券重购、以及债务重新协商的观点。

Longstaff and Schwartz (1995) 把之前研究的特色结合为一个新的模型。跟 Merton (1974) 一样, Longstaff and Schwartz 也假设公司价值会遵守随机程序。如同 Black and Cox (1976) 的做法, 他们也容许有违约风险的债券在到期日之前就可能发生违约的情形。当公司价值降到某种底线时就会发生违约的情形。正如在 Kim 等 (1987) 的模型中, 他们假设无违约风险利率会遵守 Vasicek (1977) 程序, 而且和公司价值程序也具有相关性。他们的模型假设公司的资本结构和公司价值程序没有相关性, 所以允许公司资本结构是由不同的风险或有索取权事件所构成, 包括不同债券票面息、优先选择权、和到期日。他们把这个架构用来估价债券票面息, 并且推导出一个封闭形式的算式来估价风险浮动利率债务。他们的研究显示这个模型中的信用差异和实际信用差异的许多特性是一致的。有一点重要的发现是, 如果公司资产跟利率变化有不同的相关性, 对于有相同违约风险的公司来说, 他们的信用差异具重大差异。同时, 高收益债券的特性跟低风险债务也大不相同。他们宣称其模型可随时再扩展, 以纳入和经营风险及利率风险无关的非系统性随机回复比率, 这样便可以容许优先选择权所产生的误差。Pierides (1997) 认为衍生商品的定价要保障债券持有人的债券价值不会因为信用质量的败坏而减少。他按 Merton (1974) 及 Black and Cox (1976) 制订的模型, 把衍生商品界定为债券的卖权或两个债券之间的差异的买权。他用分析性方法和数值方法导出了这些选择权的定价特性。

(二) 简约式模型

简约式模型不会明确地用违约来决定公司的价值, 实际上采用这个模型时也不需要使用和公司价值相关的参数。此模型直接处理违约程序, 而且把这个和利率结构以及回复比率结合在一起, 在违约时可用以评价风险债券。这个模型可以应用在各种违约情况, 因此较结构模型更普及。

Duffee (1999) 把公司违约的实时机率当作开方的随机程序。利用个别公司债券价格的时间序列以及横断面的特性来估算这些程序中的参数。结果显示实时违约风险的单变量模型无法切合实际公司债券回报差异的所有重要特性，特别是回报差异比较平坦或比较陡的时候。这种模型无法解释在不同公司不同信用质量所看到的信用差异的利率结构，这个可能是由于违约机率和利率中不正确的统计项所造成的，也可能是因为这个模型无法并入某些违约的特性所造成的。Duffee and Huang (1996) 运用简约模型来评价有违约风险的互换，在这过程中互换双方的信用质量可能是不对称的。他们的模型也包含了违约时结算给付的特色。他们显示了在一般利率互换时双方违约特性的不对称性程度对于决定交换利率并不是非常重要的，不对称性对于信用风险的影响在有本金交换的固定对固定支付的货币互换时是比较大的。Jarrow 等 (1997) 提出了用马可夫连锁模型来评价风险债券，并把公司的信用评级视作违约可能性的指标。这个模型提供了无套利风险的信用风险差异的利率结构，适用于信用衍生商品的定价。这个模型也利用从历史数据中所计算出来的信用评级改变以及违约的机率来定价有违约风险的债券。模型中的参数是从信用报告中的资料估算得来的，例如穆迪特别报告，或是标准普尔的信用评估。他们符合无违约风险和有违约风险债券的定价，同时也利用了从一个信用评级到另一个信用评级的历史机率。在风险中立的情况下，违约或是信用评级改变的机率是把历史机率跟违约风险溢价的因素相乘得来的。用这些在风险中立情况下的机率，我们可以评价其它有违约风险的金融工具，例如信用差异的选择权和信用违约的互换。他们假设在不同的信用评级间移动的违约风险溢价是相同的，而且信用差异只在信用评级改变时才会改变。此外，在他们的模型中违约机率和利率高低的相关性是不存在的。Lando (1998) 把违约机率和利率高低的相关性并入模型中，因此他的评价架构中有许多利率结构。然而，使用违约和信用评级改变的历史机率乃假设违约和信用评级改变的风险溢价为零。Duffee and Singleton (1999) 没有采用信用评级组合就导出了违约程序的模型。他们假设违约是由 Poisson 程序所控制的，而且短期违约机率和违约可能性是成比例的。违约机率可能会随着时间以及利率而改变，回复率也可能是随机的或是随着零票面息风险债券的市场价值而变动。在他们的实证研究中，他们假设调整违约风险后的利率会遵守开方程序 (Cox 等, 1985)，他们期后发现该模型非常符合利率交换的数据。他们的模型能够涵盖不同的无违约风险利率模型，因此也用了这些模型的评价结果。Kijima (1998) 解释了马可夫连锁模型如何导出众所周知的实证发现，例如公司过去的信用评级对于未来评级的改变有预测的作用。一家信用评级低 (高) 的公司，随着存活时间的推移，信用评级有可能会被提升 (降低)。他的模型也说明了一些实际上可信的论述，例如债券的价格和信用差异会依据其信用质量排列。Kijima and Komoribayashi (1998) 对于因为在一段时间内违约可能性低的高评级债券所产生的数值方面的问题提出了新的风险溢价调整方式。他们的模型在回复率方面是很健全的，尤其是对高评级债券而言。Arvanitis 等 (1999) 扩展 Jarrow 等 (1997) 的模

型而发展出信用差异模型，把跳跃和连续的要素都考虑进来。他们也采纳 Longstaff and Schwartz 模型中的一些特点，因为信用差异可能遵守回归均值的随机程序。他们容许转换的机率跟随某些变量从一个信用评级到另一个信用评级，也考虑到一些特别的情况可以明确的计算出信用差异。他们提供了一些按真实数据计算出来的配适曲线的例子，并且显示了这些曲线会随着时间而变化。他们也指出信用评级改变的记忆可以合并成为等级，并且说明了为什么这个模型比较适合市场价格。他们也把这模型和随机信用差异一起扩展，并且测试了转换矩阵数值和时间一致的这个基本假设，并且估计了建立信用差异模型所需的维度。

各种不同模型的比较：

模型	优点	缺点
Merton (1974)	简单，且凭直觉。	<ol style="list-style-type: none"> 1. 违约只有在到期时才会发生。 2. 只有在公司资产全部用尽时才会发生违约。 3. 无法利用过去信用评级变更以及违约的信息。 4. 需要输入公司价值。 5. 假设利率固定不变。 6. 以绝对优先权原则分配资产。
Black and Cox (1976)	和与净值或现金流量破产一致。	除了第二点之外同上。
Longstaff and Schwartz (1995)	<ol style="list-style-type: none"> 1. 并入了违约风险和利率风险，而且容许随机利率结构。 2. 模型可以加以延伸涵盖随机回复率，以容许偏离绝对优先权原则。 	除了第三、四、六点之外同上。
Jarrow 等 (1997)	<ol style="list-style-type: none"> 1. 符合观察所得的无违约风险和违约风险债券的价格，并利用从一个信用评级到另一个信用评级的历史机率获取风险中立机率。 2. 适合信用衍生商品的定价和避险。 	<ol style="list-style-type: none"> 1. 假设不同信用评级的风险回报是一样的。 2. 违约机率和利率间没有相关性。 3. 在相同的评级中没有信用差异的改变。
Duffie and Singleton (1999)	<ol style="list-style-type: none"> 1. 容许违约机率和利率间有相关性存在。 2. 回复比率可能是随机的。 3. 可包容不同的无违约风险利率结构模型。 	无法利用过往信用评级变更及违约信息。

三、模型的推导

这个模型建基于一个经济体在某特定期间内 $[0, T^*]$ 。这段期间的长度为 $h > 0$ ，任何时间点 t 有些整数 k 以 $k \cdot h$ 形式表达。假设市场上没有套利存在，且有最高评级的无违约风险以及有违约风险零折价债券的交易。

(一) 远期利率程序

$f(t, T)$ 代表适用于 $(T, T+h)$ 期间远期利率的无违约风险债券， $0 \leq t \leq T \leq T^*-h$ 适用于所有时间 t, T 。当 $t = T$ 时，利率 $f(t, t)$ 称为“即期利率”或“短期利率”，以 $r(t)$ 表示。假设远期利率曲线是从以下程序演变而来的：

$$f(t+h, T) - f(t, T) = a(t, T)h + b(t, T)x\sqrt{h}$$

$a(t, T)$ 是变动项， $b(t, T)$ 是这个程序的变动性， x 是随机变量；而 $a(t, T)$ 和 $b(t, T)$ 会随着 t 而改变。

$V_0(t, T)$ 代表无违约风险零折价债券到期日 T 时间 t 的价格：

$$V_0(t, T) = \exp\left\{\sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} f(t, kh) \cdot h\right\}$$

把 $B(t)$ 定义为在时间 t 的“货币市场账户”的价值，一元的初始投资在无违约风险的短期利率将为：

$$B(t) = \exp\left\{\sum_{k=0}^{\frac{t}{h}-1} r(kh) \cdot h\right\}$$

在风险中立的情形下，等价平赌测度是以 $B(t)$ 来定义的，以 $B(t)$ 来折价的所有资产价格都会是平赌：

$$E^t\left\{\frac{V_0(t+h, T)}{B(t+h)}\right\} = \frac{V_0(t, T)}{B(t)} \Rightarrow E^t\left\{\frac{V_0(t+h, T)}{B(t+h)} \frac{B(t)}{V_0(t, T)}\right\} = 1.$$

从 $V_0(t, T)$ 和 $B(t)$ 的定义得出：

$$\frac{V_0(t+h, T)}{V_0(t, T)} = \exp\left\{-\left(\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [f(t+h, kh) - f(t, kh)] \cdot h\right) + f(t, t) \cdot h\right\}$$

$$\frac{B(t)}{B(t+h)} = \exp\{-f(t, t) \cdot h\}.$$

因此平赌的情况便为：

$$E^t \left\{ \exp \left[- \sum_{k=\frac{t}{b}+1}^{\frac{T}{b}-1} [f(t+h, kb) - f(t, kb)] \cdot b \right] \right\} = 1.$$

以 $f(t+h, kb) - f(t, kb)$ 取代无违约风险远期利率，平赌条件则变为：

$$E^t \left\{ \exp \left[- \sum_{k=\frac{t}{b}+1}^{\frac{T}{b}-1} [a(t, kb) \cdot b^2 + b(t, kb) \cdot x \cdot b^{3/2}] \right] \right\} = 1$$

$$\sum_{k=\frac{t}{b}+1}^{\frac{T}{b}-1} a(t, kb) = \frac{1}{b^2} \ln \left\{ E^t \left[\exp \left(- \sum_{k=\frac{t}{b}+1}^{\frac{T}{b}-1} b(t, kb) \cdot x \cdot b^{3/2} \right) \right] \right\}.$$

风险中立变动项 a 和变动性 b 的回溯关系可由此导出。

(二) 马可夫移转矩阵

假设有 K 个评级可能，而且评级时会提供与包含信用风险定价结构相关的所有信息。马可夫连锁模型可以表现出信用评级的动态，状态 1 代表最高的信用等级，状态 2 代表第二高的等级，以此类推，状态 K 代表最低的信用等级，而状态 $K+1$ 就代表违约。例如，在 Moody 的评级中，状态 1 代表 Aaa，而状态 K 代表 Caa。正如 Jarrow 等 (1997) 所提出的模型一样，在这个模型中，信用评级改变的机率会随着两个信用评级以及时间而改变。为简明起见， $K+1$ 也一并纳入。

$p_{ij}(t, T)$ 代表在时间 t 到 T ，从状态 i 到状态 j 的实际机率。马可夫连锁模型的 $K+1 \times K+1$ 移转矩阵从时间 t 到 $t+b$ 是由 $P_{i,t+b}$ 而来：

$$P_{i,t+b} = \begin{bmatrix} p_{11}(t, t+b) & p_{12}(t, t+b) & \dots & p_{1K}(t, t+b) & p_{1,K+1}(t, t+b) \\ p_{21}(t, t+b) & p_{22}(t, t+b) & \dots & p_{2K}(t, t+b) & p_{2,K+1}(t, t+b) \\ \vdots & \vdots & & \vdots & \vdots \\ p_{K1}(t, t+b) & p_{K2}(t, t+b) & \dots & p_{KK}(t, t+b) & p_{K,K+1}(t, t+b) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

其中， $p_{ij}(t, t+b) \geq 0$ 适用于所有 i 和 j ， $i \neq j$ ，及适用于所有 i

$$p_{i,K+1}(t, t+b) \equiv 1 - \sum_{j=1}^K p_{ij}(t, t+b)$$

这确保各可能状态移转机率之和为 1。我们还假设违约状态 $K+1$ 遭合并，则公司一旦到达违约状态 $K+1$ ，公司维持在原来状态的机率等于 1（例：公司移转到其它状态的机率为零）。

$q_{ij}(t, T)$ 是 $p_{ij}(t, T)$ 的风险中立对应项。按风险中立机率测量，从下列算式得出从时间 t 到 $t+h$ 的对应等价平赌移转矩阵：

$$Q_{t,t+h} = \begin{bmatrix} q_{11}(t, t+h) & q_{12}(t, t+h) & \dots & q_{1K}(t, t+h) & q_{1,K+1}(t, t+h) \\ q_{21}(t, t+h) & q_{22}(t, t+h) & \dots & q_{2K}(t, t+h) & q_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ q_{K1}(t, t+h) & q_{K2}(t, t+h) & \dots & q_{KK}(t, t+h) & q_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

其中， $q_{ij}(t, t+h) \geq 0$ 适用于所有 i 和 j ， $i \neq j$ ，及适用于所有 i

$$q_{i,K+1}(t, t+h) \equiv 1 - \sum_{j=1}^K q_{ij}(t, t+h)$$

倘 $p_{ij}(t, t+h) > 0$ ，则 $q_{ij}(t, t+h) > 0$ 。为了在这些机率上加进更多结构，假设风险溢价调整项 $\theta_i(t)$ 在平赌机率的情况下，信用评级程序会符合²：

$$q_{ij}(t, t+h) = \begin{cases} \theta_i(t) p_{ij}(t, t+h) & j \neq K+1 \\ [1 - \theta_i(t)(1 - p_{i,K+1}(t, t+h))] & j = K+1 \end{cases}$$

Jarrow 等（1997）模型的溢价调整项分母中有很小的违约机率 $p_{i,K+1}$ ，为了克服这个缺点，Kijima 及 Komoribayashi（1998）提出了风险溢价调整项。³

在文献探讨中，Jarrow 等（1997）及 Kijima 和 Komoribayashi（1998）都假设风险溢价调整属决定型，即从一个信用评级到另一个信用评级的机率属

² 风险中立机率的总和是 1。

$$\begin{aligned} \sum_{j=1}^{K+1} q_{ij}(t, t+h) &= \sum_{j=K+1} \theta_i(t) \cdot p_{ij}(t, t+h) + q_{i,K+1}(t, t+h) \\ &= \theta_i(t) \cdot [1 - p_{i,K+1}(t, t+h)] + q_{i,K+1}(t, t+h) = 1 \end{aligned}$$

³ 在 Jarrow 等（1997）的模型中，风险溢价调整项是由 π 而来， $q_{ij}(t, t+h) = \pi_i(t) p_{ij}$ 适用于 i 和 j ， $i \neq j$ ，其中 $\pi_i(t)$ 是时间的决定性函数， $q_{ij}(t, t+h) \geq 0$ 适用于所有 i 和 j ， $i \neq j$ ，以及 $\sum_{j=1, j \neq i}^{K+1} q_{ij}(t, t+h) \leq 1$ 适用于 $i = 1, \dots, K+1$ ，风险溢价调整项 $\pi_i(t)$ 必须满足右列条件： $0 < \pi_i(t) \leq 1/(1 - q_{ii})$ 适用于 $i = j$ 。反之， $\pi_i(t) = [V_0(t, t+h) - V_i(t, t+h)] / [(1 - \delta) V_0(t, t+h) q_{i,K+1}]$ 。在实证研究时， $q_{i,K+1}$ 值通常都非常小，尤其是信用评级达高水平时，在这情况下，与 $V_0(t, t+h) - V_i(t, t+h)$ 比较，便不能符合这个条件。

决定型。面对不断改变的市场，参与者随时间需求不同的补偿来承受违约风险。直觉上可把风险溢价调整项变成随机来解决这个问题，这样风险中立的违约机率可能会随着这些随机变量而改变。因为短期风险中立违约机率和短期信用差异有很密切的关系，随机风险溢价调整项也比 Jarrow 等（1997）或其他模型能够容纳更多的信用差异实际模式⁴。就算信用评级不改变，信用差异也可以改变。信用差异不断改变可能是因为风险溢价的改变或流动性的影响。

（三）风险溢价调整程序

把 Jarrow 等（1997）的模型加以延伸，考虑信用差异的跳跃及连续改变，在已知的评级中信用差异是不变的。因此，信用差异及风险中立机率是由目前的信用评级和其它变量决定的。假设信用评级为 i 的有违约风险债券的风险溢价调整项，到期日为 T ，在时间 t 会随以下程序演进：

$$\theta_i(t+h, T) - \theta_i(t, T) = \alpha_i(t, T) \cdot h + \sigma_i(t, T) \cdot y \cdot \sqrt{h},$$

其中， $\alpha_i(t, T)$ 为移动项， $\sigma_i(t, T)$ 为程序的变动性， y 是随机变量。 $\alpha_i(t, T)$ 和 $\sigma_i(t, T)$ 会依据在 t 时间的所有信息而变动。 x 和 y 的相关系数是 ρ 。不同的 $\alpha_i(t, T)$ 和 $\sigma_i(t, T)$ 让我们计算出不同的风险溢价调整程序，从而为计算信用差异提供一个最合适的模型。

因为所有的机率都大于或等于 0，而且小于或等于 1，故 $\theta_i(t)$ 必须满足以下条件：

$$0 \leq \theta_i(t) \leq \frac{1}{1 - p_{i, k+1}(t, t+h)}.$$

然而，我把 $\theta_i(t) = 0$ 的情形予以排除。

$\eta_i(t, T)$ 表示信用评级为 i 的有违约风险债券的远期利率， $s_i(t, T)$ 代表 $f(t, T)$ 和 $\eta_i(t, T)$ 之间的远期信用差异。 $s_i(t, T) = \eta_i(t, T) - f(t, T)$ ⁵。 $V_i(t, T)$ 代表信用评级为 i ，到期日为 T ，在时间 t 的有违约风险零折现债券的价格：

$$V_i(t, T) = \exp \left\{ \sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} \eta_i(t, kh) \cdot h \right\}.$$

⁴ 从信用差异行为的实证研究发现，特定的风险债券的信用差异通常都会有跳跃和连续的改变。跳跃改变反应出信用变动及违约，如信用质量的不连续改变。反之，连续改变的部分指出就算无违约风险利率维持不变，在某个信用等级的债券信用差异也可能会改变。

⁵ 违约风险债券的远期利率程序可以用 $\eta_i(t, T) = f(t, T) + s_i(t, T)$ 计算。

在风险中立情况下，等价平赌测度的定义使得所用资产的价格以 $B(t)$ 折现都属平赌，这和从无违约风险债券所导出来的结果相类似：

$$E^r \left\{ \frac{V_i(t+b, T)}{B(t+b)} \right\} = \frac{V_i(t, T)}{B(t)}.$$

α_i 可当作 σ_i 及 b 的函数，风险溢价调整程序，以及无违约风险的远期利率程序。

决定远期信用差异除了违约可能性之外，还有一个很重要的因素，就是回复率。回复率是零息有违约风险债券在违约时付给索权人市场价值的部分⁶。 $\delta_i(t)$ 代表信用评级为 i 的有违约风险债券在时间 t 的回复比率。 $\delta_i(t)$ 可能包含了这个模型到时间 t 这段期间的所有信息。

对于信用评级为 i 的有违约风险债券的某段期间投资在时间 $t+b$ 时将会收到一元（倘没有违约时为 $t+b$ ，但若有违约变为 $\delta_i(t)$ ）。在风险中立时，在无违约风险利率的现金流量贴现必须和最初的投资价值相等，有违约风险债券在时间 t 的价格。无套利条件为：

$$\begin{aligned} V_i(t, t+b) &= V_0(t, t+b)[1 - q_{i, K+1}(t, t+b) + \delta_i(t) \cdot q_{i, K+1}(t, t+b)] \\ &= \exp(-f(t, t) \cdot b)[1 - q_{i, K+1}(t, t+b) + \delta_i(t) \cdot q_{i, K+1}(t, t+b)]. \end{aligned}$$

根据有违约风险债券价格的定义，有违约风险债券的价格在时间 t 、信用评级为 i 、到期日为 $t+b$ 时为：

$$V_i(t, t+b) = \exp\{-[f(t, t) + s_i(t, t)] \cdot b\}.$$

信用评级为 i 的远期信用差异与风险中立违约可能性之间的关系为：

$$s_i(t, t) = -\frac{1}{b} \ln[1 - q_{i, K+1}(t, t+b) + q_{i, K+1}(t, t+b) \cdot \delta_i(t)].$$

为了把风险溢价调整项与无违约风险及有违约风险债券的价格连结在一起，实际违约可能性和风险中立之间的转换为：

$$q_{i, K+1}(t, t+b) = 1 - \theta_i(t)(1 - p_{i, K+1}(t, t+b)).$$

反之，从无套利条件可得：

$$q_{i, K+1}(t, t+b) = \frac{V_0(t, t+b) - V_i(t, t+b)}{V_0(t, t+b) \cdot (1 - \delta_i(t))}.$$

⁶ 在此我采用 Duffie and Singleton (1999) 的“市场价值回复”而非“资金回复”（专门术语详见 Duffie and Singleton）。具相同到期日的有违约风险和无违约风险的零折现债券交易价格在违约时是相同的。

风险溢价调整 $\theta_i(t)$ 为：

$$\theta_i(t) = \frac{1}{1 - p_{i,K+1}(t, t+b)} \left\{ 1 - \frac{V_0(t, t+b) - V_i(t, t+b)}{V_0(t, t+b) \cdot (1 - \delta_i(t))} \right\}.$$

(四) 模型延伸

只要指定 $K+1 \times K+1$ 矩阵，这个模型就可以延伸为时间连续模型，这样的移转有时间一致性的马可夫连锁模型⁷：

$$\Lambda_{t,t+b} = \begin{bmatrix} \lambda_{11}(t, t+b) & \lambda_{12}(t, t+b) & \dots & \lambda_{1K}(t, t+b) & \lambda_{1,K+1}(t, t+b) \\ \lambda_{21}(t, t+b) & \lambda_{22}(t, t+b) & \dots & \lambda_{2K}(t, t+b) & \lambda_{2,K+1}(t, t+b) \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{K1}(t, t+b) & \lambda_{K2}(t, t+b) & \dots & \lambda_{KK}(t, t+b) & \lambda_{K,K+1}(t, t+b) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

其中， $\lambda_{ij}(t, t+b) \geq 0$ 适用于 i 和 j ，及适用于 i

$$\lambda_{i,K+1}(t, t+b) = - \sum_{j=1}^K \lambda_{ij}(t, t+b).$$

$K+1 \times K+1$ 在 t 期间的机率移转矩阵为：

$$P(t) = \exp(t \cdot \Lambda) = \sum_{k=0}^{\infty} \frac{(t \cdot \Lambda)^k}{k!}.$$

因此，根据 Jarrow 等（1997）所提出的步骤可建立时间连续模型，并包含随机项 $U(t)$ 。

此外，这个模型也可以扩展到不同的随机风险溢价调整程序。按相同步骤，这个模型还可涵盖回归均值或其它特点。

四、模型的建立

实证移转矩阵 P 可以轻易从过往信用等级的改变和评级报告得到，例如穆迪特别报告，或标准普尔的信用评估。回复率也可以从以往的评级报告中得到。 $\delta(t)$ 可能会随着时间或其它变量而改变，所以在不同的时间会有不同的价值。

⁷ 详见 Jarrow 等(1997)。

时间间断的假设⁸应用在随机程序变量 x 和 y ，它们是二项式随变量，每个变量为 +1 或 -1 的机率都是 1/2。两个变量间的相关系数 ρ 在模型中是可以存在的，所以 x 和 y 的共同分布区间可以假设为：

$$(x, y) = \begin{cases} (+1, +1), & \text{with prob } (1+\rho)/4 \\ (-1, -1), & \text{with prob } (1+\rho)/4 \\ (+1, -1), & \text{with prob } (1-\rho)/4 \\ (-1, +1), & \text{with prob } (1-\rho)/4 \end{cases}$$

一般来说， ρ 可能不会等于零，它甚至可能会和二项式树一致。我们可以在每段不同的时间用不同的 ρ 值，允许 ρ 随着时间或其它的变量而改变。 ρ 可以用远期利率和风险溢价调整的历史信息来估计。

一旦这两个程序中的参数，风险中立变动项 a 和 α_i 、以及波动性 b 和 σ_i ，在任何一个时间 t 计算出来，我们就可以建立一个双二项式模型，这个双二项式模型中远期利率有四个可能价值，而风险溢价调整是从每个交叉点产生的。风险溢价调整可以从以下零折现债券公式推导出来(关于多重时间的延伸请参考附录)：

$$\theta_i(t) = \frac{1}{1 - p_{i,K+1}(t, t+h)} \left\{ 1 - \frac{V_0(t, t+h) - V_i(t, t+h)}{V_0(t, t+h) \cdot (1 - \delta_i(t))} \right\}$$

另外，倘具有远期利率、风险溢价调整项，以及实证移转矩阵，我们可以用下面的公式来计算等价平赔移转矩阵，包括风险中立的违约机率和远期信用差异⁹：

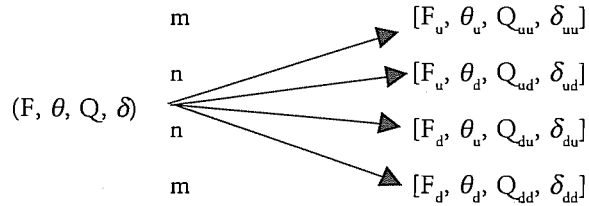
$$q_{ij}(t, t+h) = \begin{cases} \theta_i(t) p_{ij}(t, t+h) & j \neq K+1 \\ 1 - \theta_i(t) (1 - p_{i,K+1}(t, t+h)) & j = K+1 \end{cases}$$

$$s_i(t, t) = -\frac{1}{b} \ln \{ 1 - q_{i,K+1}(t, t+h) + q_{i,K+1}(t, t+h) \cdot \delta_i(t) \}$$

⁸ Das and Sundaram (2000)也采用这个假设。

⁹ 关于使用实证移转矩阵及风险溢价调整矩阵计算等价平赔矩阵的程式请参考附录。

所有和利率、违约机率、回复率等三个重要因素相关的信息都包含在风险债券评估之中，而双二项式树就变成：



$$m = \frac{1+\rho}{4} \text{ 及 } n = \frac{1-\rho}{4}.$$

把远期信用差异并入模型中，双二项式树就包含了一切厘定信用衍生商品价格的所需信息。我们可以用以下几点来解释：

1. F_u 和 F_d 分别表示当 $x = +1$ 及 -1 时 F 的远期利率。
2. θ_u 和 θ_d 分别表示当在所有信用评级 $y = +1$ 及 -1 时， θ 的风险溢价调整。¹⁰
3. Q_{uu} 指等价平赌转移矩阵（包含风险中立违约机率），已知所用信用评级的 (F_u, θ_u) ； Q_{ud} ， Q_{du} ，和 Q_{dd} ，也是用类似的方式定义。
4. δ_{uu} 指回复率，已知所用信用评级的 (F_u, θ_u) ， $(\delta_{i,uu})$ ； δ_{ud} ， δ_{du} 和 δ_{dd} ，也是用类似的方式定义。

五、模型的应用¹¹

在前面所介绍的模型可以应用到学术及实证的不同领域。最明显的应用就是在信用相关衍生商品的定价。在本节中将会分析两个例子：信用差异选择权和信用互换。除了信用衍生商品的定价之外，这个模型也可以应用在评估濒临违约公司与计算风险值（VAR）。

¹⁰ 信用评级为 i ， θ_i 的高低状态。虽然不同信用评级的有违约风险债券的风险溢价调整是由相同的变量导出（相同的随机程序），从不同信用评级的有违约风险零折现债券的历史价格所得到的不同参数，对于不同的信用评级会产生不同的风险溢价调整（排列效应 scaling effect）。

¹¹ 有关模型的应用乃按 Das 及 Sundaram（1998）改良而来。

(一) 信用差异选择权

信用差异选择权是标的物为信用差异的信用衍生商品。当一个有风险债券远期信用差异的美式买权信用在时间 t 评级为 i 的时候，买权就定义为在到期日 T 时，执行价格为 X 的到期价值。这个衍生商品可以用来确保有价证券的信用质量不会下跌。对于卖方来说它也是有价值的，因为信用差异比利率的波动性大，所以导致大量的选择权溢价。Das (1995) 也指出差异波动性接近到期日时会快速减少，时间流逝便成为吸引所有权卖方的一个特点。合约的清偿为：

$$\text{Max}[0, s_{z_T}(T, T) - X] = \{s_{z_T}(T, T) - X\}_+,$$

z_t 和 z_T 分别代表在时间 t 和 T 的信用评级，而 $z_t = i$ 。
选择权在时间 t 的价值（溢价）为：

$$\Pi_i(t, T) = E_t^Q \left\{ \exp \left[\sum_{k=\frac{t}{b}}^{\frac{T-1}{b}} f(t, kb) \cdot b \right] \cdot E_T^Q \{s_{z_T}(T, T) - X\}_+ \right\},$$

E_t^Q 代表在时间 t 双二项式树中风险中立机率 m 和 n 的期望值， E_T^Q 代表在时间 T 而 $z_T = i$ 时，在等价平赌转移矩阵 $Q_{t,T}$ （包括风险中立违约机率）在每一个分枝的期望值。

从 E_T^Q 期望值的定义以及信用差异和风险中立违约机率的关系来看， $E_T^Q \{s_{z_T}(T, T) - X\}$ 可以进一步简化为：

$$\begin{aligned} E_T^Q \{s_{z_T}(T, T) - X\}_+ &= \sum_{j=1}^{K+1} q_{ij}(t, T) \{s_j(T, T) - X\}_+ \\ &= \sum_{j=1}^{K+1} q_{ij}(t, T) \left\{ -\frac{1}{b} \ln \{1 - q_{j, K+1}(T, T+b) + q_{j, K+1}(T, T+b) \cdot \delta_j(T)\} - X \right\}_+ \end{aligned}$$

衍生商品的订价并不困难。最后的到期价值 $E_T^Q \{s_{z_T}(T, T) - X\}$ 是由分枝得来的，那就是即期差异和选择权的执行价格的差价。到期价格的折现就是说选择权价值。

(二) 信用互换

信用互换是在发生与信用有关的事件时（如违约）付给买家的一定或有索取额。或有索取额通常是指在发生信用事件时债券的票面价值和市场价值的差价，且在发生信用事件时支付。买家通常支付年金直至信用事件或是信用互换的到期日，以先发生者为准。本研究所提出的模型可以处理不同的信用事件如

信用评级改变，¹²信用互换就是一个例子。为了方便说明，假设一次过给付购买有违约风险债券的违约保险，信用评级为 i ，到期日 T^* ，信用违约互换到期日 ($T^* > T$) 违约到期价值就是债券损失的价值。若在标的债券到期前有 $(T-t)/h$ 段时间，违约可能发生在任何一段时间。在样本路径上每一点的信用违约互换到期价值为 $1 - \delta(t)$ 。到期价值因为是“第一次穿越”违约机率而增加，这个专有名词为 Das (1995) 所提出，意指之前没有违约记录。信用互换价值为：

$$\begin{aligned}
 & E_t^Q \left\{ \exp[-f(t, t) \cdot h] \cdot q_{i, K+1}(t, t+h) \cdot \{1 - \delta_i(t+h)\} + \right. \\
 & \exp\left[-\sum_{k=0}^1 f(t, t+kh) \cdot h\right] \cdot \left\{ \sum_{j=1}^K q_{i,j}(t, t+h) \cdot q_{j, K+1}(t+h, t+2h) \cdot \{1 - \delta_j(t+2h)\} \right\} + \\
 & \exp\left[-\sum_{k=0}^2 f(t, t+kh) \cdot h\right] \cdot \left\{ \sum_{j=1}^K q_{i,j}(t, t+2h) \cdot q_{j, K+1}(t+2h, t+3h) \cdot \{1 - \delta_j(t+3h)\} \right\} + \\
 & \dots + \exp\left[-\sum_{k=\frac{T-t}{h}}^{T/h-1} f(t, t+kh) \cdot h\right] \cdot \left\{ \sum_{j=1}^K q_{i,j}(t, T-h) \cdot q_{j, K+1}(T-h, T) \cdot \{1 - \delta_j(T)\} \right\} \left. \right\}
 \end{aligned}$$

如前所述， E_t^Q 代表在时间 t 双二项式树中风险中立机率 m 和 n 的期望值。我们可以扩展这个模型，以决定不同信用互换的价值。

六、结论

本研究扩展 Jarrow 等 (1997) 的模型，把随机远期利率和随机信用溢价调整并入这个模型中。这个模型对于信用差异动态的描述和历史信息的运用应能发挥更佳作用。这个模型可以容许更多的变量，也省略了不必要的假设。因此这个模型对有违约风险的公司债券和信用衍生商品的定价更为正确及更具效能。然而，我们还需要进行更多的实证研究来了解这个模型的表现，以及和其它模型的比较。

日后采用信用风险模型时还需解决一些问题。财务重整通常在违约时发生，如重新协商债务合约条款，包括延长到期日、减少或延迟到期付款、改变债务形式等等，这些在采用信用风险模型时都应予以考虑。事实上，市场会把预期债务重整的定价纳入有违约风险债券内。此外，和政府债券不同的是，某些有违约风险的公司债券甚少进行交易，故须把流动性溢价并入这些证券的定价模型内。

¹² 有些信用衍生商品，如受信用影响的票据和某些有信用反应的信用互换，它们的回报是依据特定的信用事件而定，如信用评级改变。

附录

实证移转矩阵 P 和等价平赔移转矩阵 Q 为：

$$P_{i,t+h} = \begin{bmatrix} p_{11}(t, t+h) & p_{12}(t, t+h) & \dots & p_{1K}(t, t+h) & p_{1,K+1}(t, t+h) \\ p_{21}(t, t+h) & p_{22}(t, t+h) & \dots & p_{2K}(t, t+h) & p_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ p_{K1}(t, t+h) & p_{K2}(t, t+h) & \dots & p_{KK}(t, t+h) & p_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$Q_{i,t+h} = \begin{bmatrix} q_{11}(t, t+h) & q_{12}(t, t+h) & \dots & q_{1K}(t, t+h) & q_{1,K+1}(t, t+h) \\ q_{21}(t, t+h) & q_{22}(t, t+h) & \dots & q_{2K}(t, t+h) & q_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ q_{K1}(t, t+h) & q_{K2}(t, t+h) & \dots & q_{KK}(t, t+h) & q_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

我们可以把它写成 A ， B ，和 O 等形式如下：

$$P_{i,t+h} = \begin{pmatrix} A_{i,t+h}^P & B_{i,t+h}^P \\ O & 1 \end{pmatrix} \text{ 及 } Q_{i,t+h} = \begin{pmatrix} A_{i,t+h}^Q & B_{i,t+h}^Q \\ O & 1 \end{pmatrix},$$

A^P 和 A^Q 是替代矩阵， B^P 和 B^Q 是行向量，违约机率分别是 $p_{i, K+1}$ 和 $q_{i, K+1}$ ， O 是零列向量。

$$A_{i,t+h}^P = \begin{bmatrix} p_{11}(t, t+h) & p_{12}(t, t+h) & \dots & p_{1,K-1}(t, t+h) & p_{1,K}(t, t+h) \\ p_{21}(t, t+h) & p_{22}(t, t+h) & \dots & p_{2,K-1}(t, t+h) & p_{2,K}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ p_{K-1,1}(t, t+h) & p_{K-1,2}(t, t+h) & \dots & p_{K-1,K-1}(t, t+h) & p_{K-1,K}(t, t+h) \\ p_{K,1}(t, t+h) & p_{K,2}(t, t+h) & \dots & p_{K,K-1}(t, t+h) & p_{K,K}(t, t+h) \end{bmatrix}$$

$$A_{i,t+h}^Q = \begin{bmatrix} q_{11}(t, t+h) & q_{12}(t, t+h) & \dots & q_{1,K-1}(t, t+h) & q_{1,K}(t, t+h) \\ q_{21}(t, t+h) & q_{22}(t, t+h) & \dots & q_{2,K-1}(t, t+h) & q_{2,K}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ q_{K-1,1}(t, t+h) & q_{K-1,2}(t, t+h) & \dots & q_{K-1,K-1}(t, t+h) & q_{K-1,K}(t, t+h) \\ q_{K,1}(t, t+h) & q_{K,2}(t, t+h) & \dots & q_{K,K-1}(t, t+h) & q_{K,K}(t, t+h) \end{bmatrix},$$

$\Psi_D(t)$ 代表对角线矩阵，风险溢价调整为 θ_i ， E 为由 1 构成的行向量。

$$A_{i,t+b}^Q = \Psi_D(t) \cdot A_{i,t+b}^P \quad \text{及} \quad B_{i,t+b}^Q = E - \Psi_D(t) \cdot B_{i,t+b}^P \cdot E$$

依据 Kijima and Komoribayashi (1998) 的步骤，若 $A_{0,t}^Q$ 为可逆转的，多期间风险溢价调整可以由以下方式导出：

$$A_{0,t+b}^Q = A_{0,t}^Q \cdot \Psi_D(t) \cdot A_{i,t+b}^P$$

$$\Psi_D(t) \cdot A_{i,t+b}^P \cdot E = A_{0,t}^{Q^{-1}} \cdot A_{0,t+b}^Q \cdot E$$

B_D 代表由对角线要素 $1 - q_{i,K+1} > 0$ 及 $\Psi(t) = \Psi_D(t) E$ 所构成的对角线矩阵。因此：

$$\Psi(t) = B_D^{-1} \cdot A_{0,t}^{Q^{-1}} \cdot A_{0,t+b}^Q \cdot E$$

$q_{ij}^{-1}(0, t)$ 代表 $A_{0,t}^Q$ 的构成要素：

$$\theta_i(t) = \frac{1}{1 - p_{i,K+1}(t, t+b)} \sum_{j=1}^K q_{ij}^{-1}(0, t) \cdot \frac{V_j(0, t+b) - \delta_j(t+b)V_0(0, t+b)}{[1 - \delta_j(t+b)]V_0(0, t+b)}$$

$A_{0,0}^Q = I$ (单位矩阵)。

特别是当 $t = 0$ 及 $b = 1$ ，

$$\theta_i(0) = \frac{1}{1 - p_{i,K+1}(0, 1)} \cdot \frac{V_j(0, 1) - \delta_j(1)V_0(0, 1)}{[1 - \delta_j(1)]V_0(0, 1)}$$

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A MARKOV CHAIN MODEL FOR VALUING CREDIT RISK

Shih-chuan Tsai¹

ABSTRACT

The purpose of this paper is to extend Jarrow *et al.*'s (1997) model to include stochastic default-free interest rates and stochastic credit spreads. Credit spread processes are not identified directly by the assumptions of some well-known dynamic but through the specification on the stochastic process of risk premium adjustment. By taking into account the stochastic credit spread in each credit rating in addition to credit rating changes, the model is able to consider both continuous and jump components of credit spread curves. In this model, the credit spread is allowed to change even if the credit rating remains the same. There are four distinctive features of the model. First, the credit spreads and equivalent martingale transition matrix (including risk-neutral default probabilities) are determined by both the current credit rating and by other state variables that generate the stochastic processes. Second, under the no-arbitrage condition, the recursive representations of risk-neutral drift in the forward rate process are derived, and the implicit form of the default-risky term structure can be obtained by combining the forward rate process and the indirect form of the credit spread process. Furthermore, the model facilitates the pricing of different kinds of credit derivatives and utilises historical data on information from credit rating changes and defaults. Finally, the model can easily be developed into a continuous model and different processes of risk premium adjustment can be incorporated.

Keywords: Credit Spread, Transition Matrix, Credit Spread Option, Credit Swap

I. INTRODUCTION

Valuation models of financial securities often assume that the contractual obligations will be fulfilled. However, it is possible in practice that one party of a contract may default on its obligations. The pricing of corporate default risk has recently been receiving increasing attention in financial research. Moreover, credit derivative markets have been growing rapidly during this time. A variety of models are developed to determine the premium of credit derivatives and to describe the term

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structures of default-risky corporate bonds. The models that generalised and extended the original Merton's (1974) model are called "structural models of credit risk" because their assumptions are applied to the firm's structure. An alternative approach uses "reduced-form models", which deal directly with corporate bond term structures that can be deduced from the structural models.

The pricing of credit derivatives requires credit risk information because such products are highly sensitive to a firm's credit quality. A default-risky bond can undergo several credit rating changes before it actually defaults, and the market prices those changes. The information from past credit rating changes and defaults is therefore useful in pricing default-risky corporate bonds and credit derivatives. Jarrow *et al.* (1997) develop a Markov chain model to incorporate the credit rating as an indicator of the likelihood of default to describe the term structure of credit risk spreads.

Credit risk is both similar to and different from other risks such as interest rate risk or equity risk. Credit risk can be traded just as interest rate risk may be traded. Credit risk, however, is much less liquid than interest rate risk due in part to the size of the credit market. A second difference between interest rate risk and credit risk is that changes in credit risk often trigger the associated credit spread to "jump", particularly when the changes are caused by default. Duffee (1999) applies macroeconomic changes to describe the dynamics of credit spreads. Elton *et al.* (2001) examine corporate bond spreads empirically and measure credit spreads as a function of taxes, default risk, and systematic risk factors. Huang and Kong (2003) explore the impacts of five sets of explanatory variables (default rates, interest rate variables, equity market factors, liquidity indicators, and macroeconomic indicators) on changes in credit spreads and provide evidence that credit risk models may need to incorporate macroeconomic variables to describe the dynamics of credit spreads. The fact that creditworthiness and market risk aversion evolve over time indicates the necessity of modelling credit risk dynamics as a stochastic process.

The purpose of this paper is to extend Jarrow *et al.*'s (1997) model to include stochastic default-free interest rates and stochastic credit spreads. Credit spread processes are not identified directly by the assumptions of some well-known dynamic but through the specification on the stochastic process of risk premium adjustment. By taking into account the stochastic credit spread in each credit rating in addition to credit rating changes, the model is able to consider both continuous and jump components of credit spread curves. In this model, the credit spread is allowed to change even if the credit rating remains the same. There are four distinctive features of the model. First, the credit spreads and equivalent martingale transition matrix (including risk-neutral default probabilities) are determined by both the current credit rating and by other state variables that generate the stochastic processes. Second, under a no-arbitrage condition, the recursive representations of risk-neutral drift in the forward rate process are derived, and the implicit form of the default-risky term structure can be obtained by combining the forward rate process and the indirect form of the credit spread process. Furthermore, the model facilitates the pricing of different kinds of credit derivatives and utilises historical data on information from

credit rating changes and defaults. Finally, the model can be easily developed into a continuous model and different processes of risk premium adjustment can be incorporated.

The remainder of this paper is organised as follows. Section 2 compares different approaches to credit risk and reviews several models in the literature. Section 3 describes the forward rate process and the equivalent martingale transition matrix, and specifies the settings of the model. Section 4 discusses issues on the implementation of the model and the set up of the double binomial tree evolving over time. Section 5 applies the model to determine the value of credit spread options and credit default swaps as examples of pricing credit derivatives. Section 6 concludes the article.

II. LITERATURE REVIEW

Default risk can be decomposed into two elements: the default probability and the recovery rate. The default probability is the likelihood of a default event occurring in a given period, and the recovery rate is the proportion of payment of the bond in default. Different treatments of these two elements, together with the interaction of the default risk and the interest rate risk, stimulate different studies. Two distinct approaches to the modelling of the default risk are identifiable in the literature: structural models and reduced-form models.

2.1 Structural Models

Merton (1974) explicitly values default-risky bonds by assuming a firm has only one class of pure discount bond, and that the firm value follows a diffusion process. He views equity as a call option on the firm value and considers that the bond value is the difference between the firm value and equity value, and derives the prices of default-risky bonds using the principles of option pricing developed by Black and Scholes (1973) and Merton (1973). The valuation formula requires five inputs: the value of the firm, the face amount of the debt, the volatility of the firm's value, the yield on a default-free bond that matures at the same time, and the time to maturity of the bond. Despite its simplicity and intuitive appeal, Merton's model has many limitations. First, he assumes default occurs only at maturity of the debt, which is clearly unrealistic. Second, default is assumed to occur only when the firm exhausts its assets. Since firms usually default long before the firm's assets are exhausted, the scenario is at odds with reality. Jones *et al.* (1984) and Franks and Torous (1989) show that the assumption induces credit spreads that are much smaller than actual credit spreads. Third, there is always more than one class of debt in a firm's capital structure, so the priority structures of various debts have to be specified. Also, the framework assumes the absolute-priority rule applies to allocate assets among corporate claimants. Yet another problem for the model is that the underlying assets are often not tradable and therefore their values are not observable, which makes the application of the theory and the estimation of the relevant parameters problematic.

Geske (1977) extends the analysis of Merton (1974) to risky coupon bonds that have a finite time to maturity and discrete coupon payments. Black and Cox (1976) extend Merton's (1974) analysis to the study of safety covenants, subordination arrangements, and limits on financing. They relax the assumption that default occurs only when the firm exhausts its assets and allow default to occur when the value of the firm reaches some low threshold. Their model is more realistic in that it is consistent with either net worth or cash-flow-based insolvency. They also consider the possibility that default occurs before the maturity. Their model, therefore, is able to generate credit spreads that are more consistent with those observed in corporate debt markets. The Black and Cox model, however, still has some of the other limitations of the traditional Black-Scholes-Merton framework for valuing risky debt. They assume the interest rates are constant, which is difficult to justify in a valuation model for risky fixed-income securities. They also assume the absolute-priority rule applies to allocate assets among corporate claimants. However, recent evidence by Franks and Torous (1989, 1994); Eberhart *et al.* (1990); LoPucki and Whitford (1990); Weiss (1990); Betker (1995); and others shows that strict absolute priority is rarely upheld in distressed reorganisations.

Turnbull (1979) generalises Merton's model to include corporate tax and bankruptcy costs, and derives close-form solutions for a firm's common stock and pure discount bonds. Bhattacharya and Mason (1981) extend Merton's analysis to firm value that follows a discontinuous process and to more complex boundary conditions. Kim *et al.* (1987) develop contingent claims valuation models for corporate bonds that are capable of generating default premiums consistent with the levels observed in practice. They assume the default-free interest rate follows the process described in Vasicek (1977) and is correlated with the firm value process. They study the default risk of coupons in the presence of dividends and interest rate uncertainty, and the interaction of interest rate risk and default risk. They demonstrate that default premiums are sensitive to interest rate expectations but not to the volatility of the interest rates. They also show that the call provision has a differential effect on Treasury issues relative to corporate issues. Leland (1994) derives closed-form results for the long-term risky debt, yield spreads, and optimal capital structure, when firm asset value follows a diffusion process with constant volatility. He explicitly links the debt values and optimal leverage to firm risk, taxes, bankruptcy costs, default-free interest rates, payout rates, and bond covenants. The results explain the different behaviour of junk bonds versus investment-grade bonds, and aspects of asset substitution, debt repurchase, and debt renegotiations.

Longstaff and Schwartz (1995) combine many distinctive features of the previous studies in a single model. Like Merton (1974), they assume firm value follows a diffusion process. As in Black and Cox (1976), they allow risky debt to default before maturity date. Default happens when the firm value process reaches some low boundary from above. As in Kim *et al.* (1987), the default-free interest rate is assumed to follow the Vasicek (1977) process, and to be correlated with the firm value process. Their model assumes that a firm's capital structure is irrelevant to firm value and allows the capital structure of the firm to consist of a variety of risky

contingent claims including bonds with different coupon payments, priorities, and maturity dates. They apply their framework to value-risky discount and coupon bonds, and derive closed-form expressions for the value of risky floating-rate debt. They show that the credit spreads implied by the model are consistent with many properties of actual credit spreads. An important finding is that credit spreads for firms with similar default risk can vary significantly if the assets of the firms have different correlations with changes in interest rates. They also show that the properties of high-yield bonds can be very different from those of less risky debt. They claim that their model can easily be extended to allow for the deviations from the absolute priority rule by including unsystematic stochastic recovery rates that are uncorrelated with both business risk and interest rate risk. Pierides (1997) considers the pricing of derivatives that protect holders of corporate bonds from a reduction in their value because of deterioration in their credit quality. He structures these derivatives as either puts on the bond price or calls on the bond spread in the context of the models developed by Merton (1974) and Black and Cox (1976). He derives pricing properties of these options by both analytical and numerical methods.

2.2 Reduced-form Models

Reduced-form models do not condition default explicitly on the value of the firm and parameters related to firm value are not required to implement the model. They deal directly with the default process and combine this with the term-structure model and assumptions concerning the recovery rate in default to value risky debts. They are more general than the structural models because they can accommodate different kinds of defaults.

Duffee (1999) models a firm's instantaneous probability of default as a square-root diffusion process. The parameters of these processes are estimated by both time series and cross-sectional properties of the individual firm's bond prices. The results indicate that single-factor models of instantaneous default risk have difficulty matching all the important features of actual corporate bond yield spreads, especially for both relatively flat yield spreads and steeper yield spreads. Such models cannot explain the observed term structure of credit spreads across firms of different credit qualities, which might arise from incorrect statistical specifications of default probabilities and interest rates or from the model's inability to incorporate some of the features of default. Duffie and Huang (1996) employ the reduced-form approach to value default-risky swaps in which the credit qualities of a swap's two parties can be asymmetric. Their model also incorporates features of settlement payments upon default under some of the International Swaps and Derivatives Association (ISDA). They show that the degree of asymmetry in the default characteristics of the two parties in a plain vanilla interest rate swap is not very important in determining the swap rate and that the impact of the asymmetry in the credit risk is somewhat higher in a currency swap involving fixed-for-fixed payments in which an exchange of principals take place. Jarrow *et al.* (1997) propose a Markov chain model for valuing risky debt that explicitly incorporates a firm's credit rating as an indicator of the likelihood of default. The model provides the evolution of an arbi-

trage-free term structure of credit risk spreads, and is most appropriate for the pricing and hedging of credit derivatives. They show the way to use the probabilities of credit rating changes and defaults computed from historical data to price default-risky bonds. The parameters of the model are easily estimated using observed data available in credit reports, such as Moody's Special Report or Standard and Poor's Credit Review. They match the observed prices of default-free and default-risky bonds and use the historical probabilities of migration to other credit ratings at the same time. The risk-neutral probabilities of default or credit-rating changes are then computed by multiplying the historical probabilities by a factor that can be interpreted as a default risk premium. Equipped with those risk-neutral probabilities, one can value other default-risky financial instruments of the firm, such as credit spread options and credit default swaps. They assume that the default risk premium in moving to different credit ratings is the same, and that credit spreads change only when credit ratings change. In addition, the correlation between default probabilities and the level of interest rates is not allowed in their model. Lando (1998) incorporates the correlation between default probabilities and the level of interest rates, and allows many existing term structures embedded in the valuation framework. However, historical probabilities of defaults and credit rating changes are used on the assumption that the risk premiums due to defaults and rating changes are zero. Duffie and Singleton (1999) model the default process without reference to a credit rating scheme. They assume that default is governed by the Poisson process and the probability of default over a small time interval is proportional to the default intensity. The probability of default could be time-varying and depend on the level of interest rates, and the recovery rate could be random and depend on the market value of zero-coupon risky bonds. In their empirical work, they assume that the default-adjusted interest rate follows the square root process (Cox *et al.*, 1985) and find that their model fits the interest rate swap data very well. Their model can accommodate different default-free term structure models and thus use the valuation results of those models. Kijima (1998) explains how the Markov chain model leads to the known empirical findings such that prior rating changes carry predictive power for the direction of future rating changes and that a firm with low (high) credit rating is more likely to be upgraded (downgraded) conditional on survival as the time horizon lengthens. His model also explains in a practical way plausible statements such as that bond prices as well as credit risk spreads would be ordered according to their credit qualities. Kijima and Komoribayashi (1998) propose a new risk premium adjustment to the numerical problems owing to the fact that highly rated bonds have low default probabilities within a period of time. In extensive numerical experiments, they show that their model is robust with respect to the recovery rate, especially for highly rated bonds. Arvanitis *et al.* (1999) extend Jarrow *et al.* (1997) and develop a credit spread model that takes into account both the jump and continuous components. They also incorporate some features of Longstaff and Schwartz (1995) in that credit spreads may follow a mean-reverting diffusion process. They allow transition probabilities from one credit rating to another to depend on some state variables and consider special kinds of dependence that maintain explicit computa-

tion of credit spreads. They provide some examples of fitted curves using real data and show the evolution of these curves with time. They show how the memory in credit rating changes can be incorporated into the calibration, and illustrate how this can provide a better fit to market prices. They also extend the model using stochastic credit spreads, and test the fundamental assumption that the eigenvectors of the transition matrices are constant with time and estimate the dimensionality required to model the credit spread.

Comparison of Various Models

Model	Advantages	Drawbacks
Merton (1974)	1. Simple and intuitive.	<ol style="list-style-type: none"> 1. Default occurs only at the maturity of the debt. 2. Default occurs only when the firm exhausts its assets. 3. Information on the historical credit-rating changes and defaults cannot be used. 4. Require the firm value as input. 5. Assume interest rate is constant. 6. Absolute-priority rule applies to allocate assets.
Black and Cox (1976)	Consistent with either net-worth or cash-flow-based insolvency.	Same as above except 2.
Longstaff and Schwartz (1995)	<ol style="list-style-type: none"> 1. Incorporate both default risk and interest rate risk, and allow for stochastic term structure. 2. Can be extended to include stochastic recovery rate, which allows for the deviations from strict absolute priority. 	Same as 3, 4, and 6 above.
Jarrow <i>et al.</i> (1997)	<ol style="list-style-type: none"> 1. Exactly match the observed prices of default-free and risky bonds and use the historical probabilities of migration to other credit ratings to get risk-neutral probabilities. 2. Appropriate for the pricing and hedging of credit derivatives. 	<ol style="list-style-type: none"> 1. Risk premium in moving to different credit ratings is assumed the same. 2. No correlation between default probabilities and interest rates. 3. No credit spread change in the same rating.
Duffie and Singleton (1999)	<ol style="list-style-type: none"> 1. Allow correlation between default probabilities and interest rates. 2. Recovery rate can be random. 3. Can accommodate different default-free term structure models. 	1. Information on the historical credit-rating changes and defaults cannot be used.

III. MODEL SPECIFICATION

I develop the model in discrete time and consider an economy on a finite time interval $[0, T^*]$. The length of the period is $h > 0$, and any time-point t has the form $k \cdot h$ for some integral k . I assume that there is a full range of default-free and default-risky zero-coupon bonds traded on the markets and that the markets are free of arbitrage.

3.1 Forward Rate Process

Let $f(t, T)$ denote the forward rate on the default-free bonds applicable to the period $(T, T + h)$, where $0 \leq t \leq T \leq T^* - h$ for all t, T . In particular, when $t = T$, the rate $f(t, t)$ is called the "spot rate" or "short rate", and denoted by $r(t)$. The forward rate curve is assumed to evolve following the process:

$$f(t + h, T) - f(t, T) = a(t, T)h + b(t, T)x\sqrt{h}$$

where $a(t, T)$ is the drift and $b(t, T)$ is the volatility of the process; and x is a random variable. $a(t, T)$ and $b(t, T)$ may depend on the information available at t .

Let $V_0(t, T)$ denote the price of a default-free zero coupon bond of maturity T at time t :

$$V_0(t, T) = \exp \left\{ \sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} f(t, kh) \cdot h \right\}.$$

Define $B(t)$ as the value of "money market account" at time t , which means using a \$1 initial investment and rolling over at the default-free short rate:

$$B(t) = \exp \left\{ \sum_{k=0}^{\frac{t}{h}-1} r(kh) \cdot h \right\}.$$

In a risk-neutral world, the equivalent martingale measure is defined with respect to $B(t)$ so that all asset prices discounted by $B(t)$ will be martingale:

$$E^r \left\{ \frac{V_0(t+h, T)}{B(t+h)} \right\} = \frac{V_0(t, T)}{B(t)} \Rightarrow E^r \left\{ \frac{V_0(t+h, T)}{B(t+h)} \frac{B(t)}{V_0(t, T)} \right\} = 1.$$

From the definition of $V_0(t, T)$ and $B(t)$, I have

$$\frac{V_0(t+h, T)}{V_0(t, T)} = \exp \left\{ - \left(\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [f(t+h, kh) - f(t, kh)] \cdot h \right) + f(t, t) \cdot h \right\}$$

$$\frac{B(t)}{B(t+h)} = \exp \{ -f(t, t) \cdot h \}.$$

The martingale condition, therefore, is

$$E' \left\{ \exp \left[- \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [f(t+h, kh) - f(t, kh)] \cdot h \right] \right\} = 1.$$

Substituting the process of default-free forward rate into $f(t+h, kh) - f(t, kh)$, the martingale condition becomes

$$E' \left\{ \exp \left[- \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \left[a(t, kh) \cdot h^2 + b(t, kh) \cdot x \cdot h^{3/2} \right] \right] \right\} = 1$$

$$\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} a(t, kh) = \frac{1}{h^2} \ln \left\{ E' \left[\exp \left(- \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} b(t, kh) \cdot x \cdot h^{3/2} \right) \right] \right\}.$$

A recursive relation between risk-neutral drift a and volatility b is thus derived.

3.2 Transition Matrix

Assume there are K possible ratings and being in a given rating gives all the information relevant to the pricing structures involving credit risk. A Markov chain can now represent the credit rating dynamics. State 1 represents the highest credit class, state 2 represents the second highest, state K the lowest credit class, and state $K+1$ designates default. In Moody's ratings, for example, state 1 represents Aaa and state K represents Caa. In the model, the probabilities of credit rating changes depend on the two credit ratings and the length of the time (time-homogeneous Markov chain), as introduced by Jarrow *et al.* (1997). Assume also the default state $K+1$ is absorbing for the sake of simplicity.

Let $p_{ij}(t, T)$ denote the actual probability of going from state i at time t to state j at time T . The $(K+1) \times (K+1)$ transition matrix of the Markov chain from time t to time $t+h$ is given by $P_{t,t+h}$,

$$P_{t,t+h} = \begin{bmatrix} p_{11}(t, t+h) & p_{12}(t, t+h) & \dots & p_{1K}(t, t+h) & p_{1,K+1}(t, t+h) \\ p_{21}(t, t+h) & p_{22}(t, t+h) & \dots & p_{2K}(t, t+h) & p_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ p_{K1}(t, t+h) & p_{K2}(t, t+h) & \dots & p_{KK}(t, t+h) & p_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

where $p_{ij}(t, t+h) \geq 0$ for all i, j , $i \neq j$, and for all i

$$p_{i,K+1}(t, t+h) \equiv 1 - \sum_{j=1}^K p_{ij}(t, t+h)$$

This ensures that the probabilities of possible state transitions add up to one. Notice that the assumption of the default state $K + 1$ being absorbing implies that once a firm reaches the default state $K + 1$, the probability of the firm's staying in the same state equals one (e.g. the probability of the firm transiting to any other state equals zero).

Let $q_{ij}(t, T)$ be the risk-neutral counterpart of $p_{ij}(t, T)$. Under a risk-neutral probability measure, the corresponding equivalent martingale transition matrix from time t to time $t + h$ is given by:

$$Q_{t,t+h} = \begin{bmatrix} q_{11}(t, t+h) & q_{12}(t, t+h) & \dots & q_{1K}(t, t+h) & q_{1,K+1}(t, t+h) \\ q_{21}(t, t+h) & q_{22}(t, t+h) & \dots & q_{2K}(t, t+h) & q_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ q_{K1}(t, t+h) & q_{K2}(t, t+h) & \dots & q_{KK}(t, t+h) & q_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

where $q_{ij}(t, t+h) \geq 0$ for all i, j , $i \neq j$, and for all i

$$q_{i,K+1}(t, t+h) \equiv 1 - \sum_{j=1}^K q_{ij}(t, t+h)$$

And $p_{ij}(t, t+h) > 0$ if and only if $q_{ij}(t, t+h) > 0$. To impose more structure on these probabilities, I assume the risk premium adjustments, $\theta_i(t)$, are such that the credit rating process under the martingale probabilities satisfies²

$$q_{ij}(t, t+h) = \begin{cases} \theta_i(t) p_{ij}(t, t+h) & j \neq K+1 \\ 1 - \theta_i(t)(1 - p_{i,K+1}(t, t+h)) & j = K+1 \end{cases}$$

The risk premium adjustment is proposed by Kijima and Komoribayashi (1998) to overcome the drawback of the premium adjustment in Jarrow *et al.* (1997) due to the small default probability $p_{i, K+1}$ in the denominator.³

² The sum of the risk-neutral probability is equal to 1.

$$\begin{aligned} \sum_{j=1}^{K+1} q_{ij}(t, t+h) &= \sum_{j \neq K+1} \theta_i(t) \cdot p_{ij}(t, t+h) + q_{i,K+1}(t, t+h) \\ &= \theta_i(t) \cdot [1 - p_{i,K+1}(t, t+h)] + q_{i,K+1}(t, t+h) = 1 \end{aligned}$$

³ In Jarrow *et al.* (1997), the risk premium adjustment is given by π such that $q_{ij}(t, t+h) = \pi_i(t) p_{ij}$ for all i, j , $i \neq j$ where $\pi_i(t)$ is a deterministic function of time and $q_{ij}(t, t+h) \geq 0$ for all i, j , $i \neq j$, and $\sum_{j=1, j \neq i}^{K+1} q_{ij}(t, t+h) \leq 1$ for $i = 1, \dots, K+1$, the risk premium adjustment $\pi_i(t)$ has to satisfy the condition: $0 < \pi_i(t) \leq 1/(1 - q_{ii})$ for $i = j$. On the other hand, $\pi_i(t) = [V_0(t, t+h) - V_1(t, t+h)] / [(1 - \delta) V_0(t, t+h) q_{1,K+1}]$. When $q_{1,K+1}$ is sufficiently small, which is always true in empirical research especially for high credit ratings, compared to $V_0(t, t+h) - V_1(t, t+h)$, the condition would be violated.

In both Jarrow *et al.* (1997) and Kijima and Komoribayashi (1998), the risk premium adjustment is assumed to be deterministic. This means that the probabilities of going from one credit rating to another are deterministic. Facing the ever-changing market condition, market participants may require different compensation over time in order to bear the default risk. An intuitive way to deal with it is to make the risk premium adjustment stochastic and thus risk-neutral default probabilities may depend on some stochastic state variables. Since short-term risk-neutral default probabilities are closely related to short-term credit spreads (the relation will be derived later in this section), stochastic risk premium adjustments also allow for more realistic patterns⁴ for credit spreads than in Jarrow *et al.* (1997) or other studies. This allows changes in credit spreads even if the credit rating does not change. The continuous changes in credit spreads may be due to variations in risk premiums and liquidity effects.

3.3 Risk Premium Adjustment Process

It is natural to extend Jarrow *et al.*'s (1997) model, where the credit spreads for a given rating are constant, to take into account both the jump and continuous changes in credit spreads. Credit spreads and risk-neutral probabilities, therefore, are determined by the current credit rating and by some other state variables. Assume the risk premium adjustment of a default-risky bond with credit rating i and maturity T at time t evolves following the process

$$\theta_i(t+h, T) - \theta_i(t, T) = \alpha_i(t, T) \cdot h + \sigma_i(t, T) \cdot y \cdot \sqrt{h},$$

where $\alpha_i(t, T)$ is the drift and $\sigma_i(t, T)$ is the volatility of the process; and y is a random variable. $\alpha_i(t, T)$ and $\sigma_i(t, T)$ may depend on the information available at t . The correlation coefficient between x and y is ρ . Different specifications of $\alpha_i(t, T)$ and $\sigma_i(t, T)$ allow us to formulate various risk premium adjustment processes that can in turn provide us with the most appropriate model for credit spreads in practice.

Because all the probabilities are larger than or equal to zero, and smaller than or equal to 1, $\theta_i(t)$ must satisfy the condition

$$0 \leq \theta_i(t) \leq \frac{1}{1 - p_{i,K+1}(t, t+h)}.$$

However, I exclude the situation that $\theta_i(t) = 0$.

Let $\eta_i(t, T)$ denote the forward rate on default-risky bonds of credit rating i implied from the spot yield curve, and $s_i(t, T)$ denote the forward credit spread between $f(t, T)$ and $\eta_i(t, T)$. $s_i(t, T) = \eta_i(t, T) - f(t, T)$.⁵ Let $V_i(t, T)$ denote the price of a default-risky zero coupon bond of credit rating i and maturity T at time t :

⁴ From the empirical research on credit spread behaviour, the credit spread for a specific risky bond typically exhibits both a jump and a continuous change. The jump part may reflect credit migration and default, which is a discontinuous change of credit quality. On the other hand, the continuous part indicates that the credit spread on a bond of a given credit rating may change even if the default-free interest rates remain constant.

⁵ The forward rate process on risky bonds can be calculated by $\eta_i(t, T) = f(t, T) + s_i(t, T)$.

$$V_i(t, T) = \exp \left\{ \sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} \eta_i(t, kh) \cdot h \right\}.$$

In the risk-neutral world, similar to what is derived for the default-free bonds, the equivalent martingale measure is defined so that all asset prices discounted by $B(t)$ will be martingale:

$$E^t \left\{ \frac{V_i(t+h, T)}{B(t+h)} \right\} = \frac{V_i(t, T)}{B(t)}.$$

α_i can be derived as a function of σ_i and b , the diffusion terms of the risk premium adjustment process and the default-free forward rate process.

Besides default probability, there is one more important component in determining the forward credit spread: the recovery rate. The recovery rate is the fraction of market value that zero-coupon default-risky bonds would pay their claimholders in the event of default.⁶ Let $\delta_i(t)$ denote the recovery rate of default risky bonds with credit rating i at time t . $\delta_i(t)$ may include all information in the model up to and including period t .

A one-period investment in the default-risky bond with credit rating i pays \$1 at time $t+h$ if there is no default at $t+h$, and $\delta_i(t)$ if there is a default. In the risk-neutral world, the expected cash flow discounting at the default-free rate must equal the initial value of the investment, the price of the default-risky bond at time t . The no-arbitrage condition is

$$\begin{aligned} V_i(t, t+h) &= V_0(t, t+h)[1 - q_{i,K+1}(t, t+h) + \delta_i(t) \cdot q_{i,K+1}(t, t+h)] \\ &= \exp(-f(t, t) \cdot h)[1 - q_{i,K+1}(t, t+h) + \delta_i(t) \cdot q_{i,K+1}(t, t+h)]. \end{aligned}$$

By the definition of the default-risky bond price, the price of a default-risky bond at time t with credit rating i maturing at $t+h$ would be

$$V_i(t, t+h) = \exp\{-(f(t, t) + s_i(t, t)) \cdot h\}.$$

The relation between the forward credit spread for credit rating i and the risk-neutral default probability can be derived as

$$s_i(t, t) = -\frac{1}{h} \ln\{1 - q_{i,K+1}(t, t+h) + q_{i,K+1}(t, t+h) \cdot \delta_i(t)\}.$$

To link the risk premium adjustment to the prices of default-free and default-risky bonds, it will be recalled that the transformation of the actual default probability to risk-neutral one is

⁶ Here, I use the assumption of "Recovery of Market Value" (RMV) in Duffie and Singleton (1999) instead of the assumption of "Recovery of Treasury" (the terminology is from Duffie and Singleton) that a zero-coupon risky bond trades for the same price as the fraction (δ) of a default-free zero coupon bond with the same maturity in the event of default.

$$q_{i,K+1}(t, t+h) = 1 - \theta_i(t)(1 - p_{i,K+1}(t, t+h)).$$

On the other hand, from the no-arbitrage condition, I have

$$q_{i,K+1}(t, t+h) = \frac{V_0(t, t+h) - V_i(t, t+h)}{V_0(t, t+h) \cdot (1 - \delta_i(t))}.$$

The risk premium adjustment $\theta_i(t)$ can be obtained by the following:

$$\theta_i(t) = \frac{1}{1 - p_{i,K+1}(t, t+h)} \left\{ 1 - \frac{V_0(t, t+h) - V_i(t, t+h)}{V_0(t, t+h) \cdot (1 - \delta_i(t))} \right\}.$$

3.4 Extension

The setup can be easily extended to the continuous-time model by specifying the $K+1 \times K+1$ generator matrix for a time-homogeneous transition Markov chain:⁷

$$\Lambda_{i,t+h} = \begin{bmatrix} \lambda_{11}(t, t+h) & \lambda_{12}(t, t+h) & \dots & \lambda_{1K}(t, t+h) & \lambda_{1,K+1}(t, t+h) \\ \lambda_{21}(t, t+h) & \lambda_{22}(t, t+h) & \dots & \lambda_{2K}(t, t+h) & \lambda_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{K1}(t, t+h) & \lambda_{K2}(t, t+h) & \dots & \lambda_{KK}(t, t+h) & \lambda_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where $\lambda_{ij}(t, t+h) \geq 0$ for all i, j , and for all i

$$\lambda_{i,K+1}(t, t+h) \equiv - \sum_{j=1}^K \lambda_{ij}(t, t+h).$$

The $K+1 \times K+1$ t-period probability transition matrix is given by

$$P(t) = \exp(t \cdot \Lambda) = \sum_{k=0}^{\infty} \frac{(t \cdot \Lambda)^k}{k!}.$$

The continuous-time model can thus be developed following steps in Jarrow *et al.* (1997) and including a stochastic term $U(t)$.

In addition, the setup can be extended easily to different stochastic risk premium adjustment processes. Following the same steps, one can include mean-reversion or other features in the model.

IV. MODEL IMPLEMENTATION

The empirical transition matrix P can be easily obtained from past observations of credit rating changes and defaults in rating agencies' reports, Moody's Special Report,

⁷ For more detail, see Jarrow *et al.* (1997).

and Standard & Poor's Credit Review, for example. The recovery rate can also be obtained from historical data in those rating agencies' reports. $\delta(t)$ may change with time or depend on some state variables so that it may have different values in each period.

The standard discrete-time assumption⁸ applied to stochastic process variables, x and y , is that they are binomial random variables and that each variable takes the values of either +1 or -1 with probability 1/2. The correlation ρ between two variables is allowed in the model, so the assumed joint distribution of x and y is

$$(x, y) = \begin{cases} (+1, +1), & \text{with prob } \frac{(1 + \rho)}{4} \\ (-1, -1), & \text{with prob } \frac{(1 + \rho)}{4} \\ (+1, -1), & \text{with prob } \frac{(1 - \rho)}{4} \\ (-1, +1), & \text{with prob } \frac{(1 - \rho)}{4} \end{cases}$$

In general, ρ may not be equal to zero or may even be constant over the binomial tree. One may apply in each period with a different value of ρ and allow ρ to change with time or depend on some state variables. ρ can be estimated by historical data of forward interest rate and risk premium adjustment.

Once the parameters of the two processes, risk-neutral drifts a and α , as well as volatility b and σ_p , are computed at any time t , a double-binomial model can be constructed with four branches of possible values of forward interest rates and risk premium adjustment emanating from each node. The risk premium adjustment can be inferred from the zero-coupon bond prices by the following formula (See appendix for the multi-period extension):

$$\theta_i(t) = \frac{1}{1 - p_{i,K+1}(t, t+h)} \left\{ 1 - \frac{V_0(t, t+h) - V_i(t, t+h)}{V_0(t, t+h) \cdot (1 - \delta_i(t))} \right\}$$

Moreover, given forward interest rates, risk premium adjustment, and empirical transition matrix, one can obtain the equivalent martingale transition matrix (including risk-neutral default probabilities) and forward credit spread by the following formula⁹:

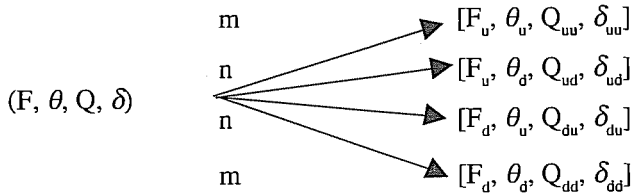
$$q_{ij}(t, t+h) = \begin{cases} \theta_i(t) p_{ij}(t, t+h) & j \neq K+1 \\ 1 - \theta_i(t)(1 - p_{i,K+1}(t, t+h)) & j = K+1 \end{cases}$$

$$s_i(t, t) = -\frac{1}{h} \ln \{ 1 - q_{i,K+1}(t, t+h) + q_{i,K+1}(t, t+h) \cdot \delta_i(t) \}$$

⁸ The same implementation is also used in Das and Sundaram (2000).

⁹ See appendix for the matrix form of computing equivalent martingale transition matrix by empirical transition matrix and risk premium adjustment matrix.

All the information relating to the three key factors (interest rate, default probabilities, and recovery rate) involved in the valuation of risky debt is therefore obtained and the double-binomial tree evolves as follows:



$$m = \frac{1 + \rho}{4} \quad \text{and} \quad n = \frac{1 - \rho}{4}.$$

Incorporating the forward credit spread, the double binomial tree involves all the necessary information for pricing a range of credit derivatives. The notation is explained as the following:

1. F_u and F_d refer, respectively, to the forward interest rates that result from F if $x = +1$ and -1 .
2. θ_u and θ_d refer, respectively, to the risk premium adjustment that results from θ if $y = +1$ and -1 for the whole range of credit ratings.¹⁰
3. Q_{uu} refers to the equivalent martingale transition matrix (including risk-neutral default probabilities) given (F_u, θ_u) for the whole range of credit ratings; Q_{ud} , Q_{du} , and Q_{dd} , are defined analogously.
4. δ_{uu} refers to the recovery rate given (F_u, θ_u) for the whole range of credit ratings ($\delta_{i,uu}$); δ_{ud} , δ_{du} , and δ_{dd} , are defined analogously.

V. APPLICATION¹¹

The model presented in the previous section can be applied to different fields of academic research and practical use. The most straightforward application is probably the pricing of credit-related derivatives. Two examples, credit spread options and credit swaps, are analysed in this section. In addition to credit derivative pricing, the model draws important implications with regard to the valuation of near-default firms and to the computation of value of risk (VAR).

5.1 Credit Spread Option

A credit spread option is a credit derivative that is written on an underlying credit spread. Consider a European call option written on the forward credit spread of a

¹⁰ The up state and down state of θ_i , for all credit rating i . Although risk premium adjustment processes of default-risky bonds with different credit ratings are generated by the same state variable (the same diffusion process), different parameters estimated from historical prices of default-risky zero-coupon bonds with different credit ratings would produce different risk premium adjustments (scaling effect) for different credit ratings in each state.

¹¹ The applications are modified from Das and Sundaram (1998).

default-risky bond with credit rating i at time t . The call option is defined as a contract that pays off at maturity date T if the spread is trading above an exercise price X . This derivative may be used to insure a bond portfolio against declines in credit quality. It is also valuable for option writers since credit spreads tend to be more volatile than interest rates, resulting in large option premiums. Furthermore, Das (1995) indicates that spread volatility declines rapidly towards bond maturity, making time decay an attractive feature for spread option writers. The payoff of the contract is

$$\text{Max}[0, s_{z_T}(T, T) - X] = \{s_{z_T}(T, T) - X\}_+,$$

where z_t and z_T represent the credit rating at time t and T respectively. In this case, $z_t = i$.

The value (the premium) of the option at time t is given by

$$\Pi_i(t, T) = E_t^Q \left\{ \exp \left\{ \sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} f(t, kh) \cdot h \right\} \cdot E_T^Q \{s_{z_T}(T, T) - X\}_+ \right\},$$

where E_t^Q denotes the expectation operator at time t with risk-neutral probabilities m and n in the double-binomial tree, and E_T^Q denotes the expectation operator at time T conditional on $z_t = i$ with the equivalent martingale transition matrix $Q_{i,T}$ (including risk-neutral default probabilities) on each lattice.

From the definition of the expectation operator E_T^Q and the relation between the credit spread and risk-neutral default probabilities, $E_T^Q \{s_{z_T}(T, T) - X\}_+$ can be further simplified:

$$\begin{aligned} E_T^Q \{s_{z_T}(T, T) - X\}_+ &= \sum_{j=1}^{K+1} q_{ij}(t, T) \{s_j(T, T) - X\}_+ \\ &= \sum_{j=1}^{K+1} q_{ij}(t, T) \left\{ -\frac{1}{h} \ln \{1 - q_{j,K+1}(T, T+h) + q_{j,K+1}(T, T+h) \cdot \delta_j(T)\} - X \right\}_+ \end{aligned}$$

Pricing this derivative is not difficult. Terminal payoffs $E_T^Q \{s_{z_T}(T, T) - X\}_+$ are generated on the lattice by taking the expectation of the positive difference between the spot spread at that time and the exercise price of the option. Discounting these payoffs back appropriately gives the option premium.

5.2 Credit Swap

Credit swaps pay the buyer a given contingent amount at the time of a given credit event, such as a default. The contingent amount is often the difference between the face value of a bond and its market value on the credit event, and is paid at the time the credit event occurs. The buyer usually pays an annuity premium until the time of credit event or the maturity date of the credit swap, whichever comes first. The

setting in this paper can easily handle different credit events such as credit rating changes.¹² Here, a credit default swap is considered as an example. Assume, for the sake of simplicity and illustration, a single lump-sum payment is made to purchase a default insurance for a default-risky bond with credit rating i and maturity T^* in the credit default swap contract maturing at time T ($T^* > T$). The payoff on default is the loss in the value on the bond. If there are $(T - t)/h$ periods before the underlying bond matures, then the default may happen in any of the periods. The payoffs of the credit default swap are $1 - \delta(t)$ at all points on the sample path. These payoffs are multiplied by the "first-passage" (the terminology is used in the Das (1995)) default probability, which is the probability of default conditional on no prior default. The value of the credit default swap is given by

$$\begin{aligned}
& E_t^Q \left\{ \exp[-f(t, t) \cdot h] \cdot q_{i, K+1}(t, t+h) \cdot \{1 - \delta_i(t+h)\} + \right. \\
& \exp\left[-\sum_{k=0}^1 f(t, t+kh) \cdot h\right] \cdot \left\{ \sum_{j=1}^K q_{i,j}(t, t+h) \cdot q_{j, K+1}(t+h, t+2h) \cdot \{1 - \delta_j(t+2h)\} \right\} + \\
& \exp\left[-\sum_{k=0}^2 f(t, t+kh) \cdot h\right] \cdot \left\{ \sum_{j=1}^K q_{i,j}(t, t+2h) \cdot q_{j, K+1}(t+2h, t+3h) \cdot \{1 - \delta_j(t+3h)\} \right\} + \\
& \dots + \exp\left[-\sum_{k=0}^{T/h-1} f(t, t+kh) \cdot h\right] \cdot \left\{ \sum_{j=1}^K q_{i,j}(t, T-h) \cdot q_{j, K+1}(T-h, T) \cdot \{1 - \delta_j(T)\} \right\} \left. \right\}
\end{aligned}$$

where E_t^Q , as above, denotes the expectation operator at time t with risk-neutral probabilities m and n in the double-binomial tree. It is not difficult to extend the pricing model to determine the values of different credit swaps.

VI. CONCLUDING REMARKS

This paper extends Jarrow *et al.* (1997) to include both the stochastic forward interest rate and the stochastic risk premium adjustment. The model is expected to better describe the dynamics of credit spreads and utilise historical information. By allowing for more variables and skipping unnecessary assumptions, the model is designed to price default-risky corporate bonds and credit derivatives more accurately and efficiently. However, there is still a lot of empirical research to be done to understand the performance of the model and the comparison to existing models.

There are some other problems that need to be solved in the future modelling of credit risk. Financial restructuring that often occurs upon defaults, such as the renegotiation of the terms of the debt contract, including the extension of the maturity,

¹² There are credit derivatives such as credit-sensitive notes and certain types of swaps (with credit triggers) whose payoffs explicitly depend on the occurrence of particular credit events, such as credit rating changes.

the reduction or delay of due payments, the change of debt form, and so on, ought to be considered in credit risk models. In practice, the market would price the anticipated debt restructuring into the value of a default-risky bond in one way or another. Moreover, unlike government bonds, some default-risky corporate bonds are thinly traded. A liquidity premium should therefore be incorporated into the pricing model of these securities.

APPENDIX

The empirical transition matrix P and the equivalent martingale transition matrix Q is

$$P_{t,t+h} = \begin{bmatrix} p_{11}(t, t+h) & p_{12}(t, t+h) & \dots & p_{1K}(t, t+h) & p_{1,K+1}(t, t+h) \\ p_{21}(t, t+h) & p_{22}(t, t+h) & \dots & p_{2K}(t, t+h) & p_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ p_{K1}(t, t+h) & p_{K2}(t, t+h) & \dots & p_{KK}(t, t+h) & p_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$Q_{t,t+h} = \begin{bmatrix} q_{11}(t, t+h) & q_{12}(t, t+h) & \dots & q_{1K}(t, t+h) & q_{1,K+1}(t, t+h) \\ q_{21}(t, t+h) & q_{22}(t, t+h) & \dots & q_{2K}(t, t+h) & q_{2,K+1}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ q_{K1}(t, t+h) & q_{K2}(t, t+h) & \dots & q_{KK}(t, t+h) & q_{K,K+1}(t, t+h) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

It can be written in the form of A , B , and O as follows:

$$P_{t,t+h} = \begin{pmatrix} A_{t,t+h}^P & B_{t,t+h}^P \\ O & 1 \end{pmatrix} \quad \text{and} \quad Q_{t,t+h} = \begin{pmatrix} A_{t,t+h}^Q & B_{t,t+h}^Q \\ O & 1 \end{pmatrix},$$

where A^P and A^Q are the sub-matrix defined on the non-absorbing states, B^P and B^Q are the column vectors respectively with default probability $p_{i,K+1}$ and $q_{i,K+1}$ as components, and O is the zero row vector.

$$A_{t,t+h}^P = \begin{bmatrix} p_{11}(t, t+h) & p_{12}(t, t+h) & \dots & p_{1,K-1}(t, t+h) & p_{1,K}(t, t+h) \\ p_{21}(t, t+h) & p_{22}(t, t+h) & \dots & p_{2,K-1}(t, t+h) & p_{2,K}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ p_{K-1,1}(t, t+h) & p_{K-1,2}(t, t+h) & \dots & p_{K-1,K-1}(t, t+h) & p_{K-1,K}(t, t+h) \\ p_{K,1}(t, t+h) & p_{K,2}(t, t+h) & \dots & p_{K,K-1}(t, t+h) & p_{K,K}(t, t+h) \end{bmatrix}$$

$$A_{t,t+h}^Q = \begin{bmatrix} q_{11}(t, t+h) & q_{12}(t, t+h) & \dots & q_{1,K-1}(t, t+h) & q_{1,K}(t, t+h) \\ q_{21}(t, t+h) & q_{22}(t, t+h) & \dots & q_{2,K-1}(t, t+h) & q_{2,K}(t, t+h) \\ \vdots & \vdots & & \vdots & \vdots \\ q_{K-1,1}(t, t+h) & q_{K-1,2}(t, t+h) & \dots & q_{K-1,K-1}(t, t+h) & q_{K-1,K}(t, t+h) \\ q_{K,1}(t, t+h) & q_{K,2}(t, t+h) & \dots & q_{K,K-1}(t, t+h) & q_{K,K}(t, t+h) \end{bmatrix},$$

$\Psi_D(t)$ is denoted to be the diagonal matrix with the risk premium adjustments θ_i as diagonal components, and E the column vector with 1 as components.

$$A_{t,t+h}^Q = \Psi_D(t) \cdot A_{t,t+h}^P \quad \text{and} \quad B_{t,t+h}^Q = E - \Psi_D(t) \cdot A_{t,t+h}^P \cdot E$$

Following the steps in Kijima and Komoribayashi (1998), if $A_{0,t}^Q$ is invertible, multi-period risk premium adjustments can be derived as follows:

$$A_{0,t+h}^Q = A_{0,t}^Q \cdot \Psi_D(t) \cdot A_{t,t+h}^P$$

$$\Psi_D(t) \cdot A_{t,t+h}^P \cdot E = A_{0,t}^{Q^{-1}} \cdot A_{0,t+h}^Q \cdot E$$

B_D is denoted to be the diagonal matrix with diagonal components $1 - q_{i,K+1} > 0$ and $\Psi(t) = \Psi_D(t) E$. So that

$$\Psi(t) = B_D^{-1} \cdot A_{0,t}^{Q^{-1}} \cdot A_{0,t+h}^Q \cdot E$$

$q_{ij}^{-1}(0, t)$ is denoted to be the components of $A_{0,t}^Q$, and writing in terms of components,

$$\theta_i(t) = \frac{1}{1 - p_{i,K+1}(t, t+h)} \sum_{j=1}^K q_{ij}^{-1}(0, t) \cdot \frac{V_j(0, t+h) - \delta_j(t+h)V_0(0, t+h)}{[1 - \delta_j(t+h)]V_0(0, t+h)}$$

with $A_{0,0}^Q = I$, the identity matrix.

In particular, when $t = 0$ and $h = 1$,

$$\theta_i(0) = \frac{1}{1 - p_{i,K+1}(0, 1)} \cdot \frac{V_j(0, 1) - \delta_j(1)V_0(0, 1)}{[1 - \delta_j(1)]V_0(0, 1)}$$

REFERENCES

Please refer to P. 18–20